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MATHEMATIC FORMALIZATION OF THE ECONOMIC SOCIAL COMMUNITARIAN PRODUCTIVE MODEL

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ABSTRACT

The purpose of this work is to mathematically formalize the Economic Social Communitarian Productive Model implemented in Bolivia since 2006. For this purpose, a mathematic model is developed based on a super-predator model, in which the solutions are obtained through Lokta-Volterra type equations. The model shows that with economic surpluses' transfers from the strategic sector toward the income and employment generating sector, it is possible to achieve sustained economic growth.

Classification JEL: C02, C61, P27
Key words: Economic Model, Dynamic Systems

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I. INTRODUCTION

a. Economic Model: Background

Subsequent to primitive communism, the history of societies and social classes can be shortened in the struggle for the appropriation of society's economic surplus. An economic model defines how the economic surplus is generated and distributed among the members of a society. An economic model involves a way to organize the production and distribution, therefore, a way to organize social relations of production. In the history of humankind there have been several economic models under different modes of production which have also established different social relations. Similarly, within the same mode of production different economic models for surplus distribution have been generated. They raised without eliminating the fundamental contradictions of these modes of production. These relations, that include the productive and the appropriation of surplus, determine the way in which societies organize themselves in the judicial, religious and cultural aspects, namely the superstructure.

In Bolivia, the neoliberal model was ruling for more than twenty years, since the enactment of the Supreme Decree N° 21060 in August 29, 1985. This model was guided under the fundaments of trade opening, the economic activities liberalization and the market as the most efficient allocator of resources. This model based the economic development on the external demand and private initiative, diminishing the participation of the State to a minimum.

But the most questioned aspect of these neoliberal measures in Bolivia was, without a doubt, the disposal of natural resources and the privatization of public enterprises under the pretext of improving efficiency. This opening to foreign capital brought the exploitation of natural resources by several transnational enterprises which established monopolies in the country.

The neoliberal model provided income concentration, it did not take into account the diverse socioeconomic reality of the country, and pre-capitalist modes of production coexistent in Bolivia. This model reduced formal employment and increased informal employment. It also increased the economic differences between social classes, deepening extreme poverty and social problems.

During the neoliberal model term, surplus production and its distribution was not equitable, as it was concentrated in few national and foreign hands.

b. Basis of the Economic Social Communitarian Productive Model

Current capitalist system faces a structural crisis never seen before, which is apparent in 7 ways: financial, energy, climatic, food, hydric, institutional and politic macroeconomic crisis.

This crisis context can become opportunities for Bolivians, inasmuch as the country has natural resources to become a great energy generator, and producer of diverse industrialized products, derivatives of gas, lithium and others.

Thus, in 2005 a new economic model emerges consistent with the country's socioeconomic reality, which was structured in the 2007 National Development Plan. This new model is called the Economic Social Communitarian Productive Model, (MESCP, Spanish acronym).

The fundamentals of the MESCP are: i) Growth and development based on the exploitation of natural sources to benefit Bolivian population, ii) State appropriation of the strategic economic sectors' economic surplus, iii) redistribution of the economic surplus toward the most vulnerable sectors, and iv) reduction of the social and poverty inequality.

In this framework, it is important that the strategic surplus generating sectors be owned by the State (nationalization process).[1] Thus, these surpluses can be redistributed to all Bolivians, especially among those most vulnerable and impoverished part of the population. It is worth noting that this is essentially a revenue redistributor model.[2]

c. Performance of the Economic Social Community Productive Model

The MESCP identifies two important sectors: i) the strategic sector, where the largest part of surplus in the country is generated, and ii) the income and employment generating sectors, which include potential economic sectors not yet been fully developed.

In this framework, the MESCP considers three strategic activities in order to generate economic surplus: hydrocarbons, mining and electricity. On the other hand, among the income and employment generating sectors are manufacturing industry, tourism, housing, agricultural development and others.

According to the MESCP, in order to develop a new productive Bolivia and to modify the primary export model, it is necessary to take all surplus from the strategic sectors towards the sectors which require the cornerstone for a productive and industrialized country, namely, to the income and employment generating sectors. The State is the responsible for performing the process of redistribution.

Although it is clear that Bolivia for a while will continue being a primary export country, the MESCP aims to free Bolivia from the export of raw materials dependency in order to abandon the primary export model and build an industrialized and productive Bolivia.

[1] The process of nationalization in Bolivia was initiated on May 1st, 2006, by the Supreme Decree N° 28701, approved by Evo Morales, whose objective was to recover the ownership and control of hydrocarbons. It is worth noting that as a result of the new regulatory framework established by the nationalization, the fiscal revenues from hydrocarbons increased by more than 100%. Afterwards, the process of nationalization was performed in other strategic sectors such as mining, telecommunication and electricity.

[2] The revenue redistribution policies, in the framework of the MESCP, are the conditional cash transfers (Juancito Pinto, Dignity Rent and Juana Azurduy), salary increases higher than the inflation rate, the increase of national minimum wages, cross-subsidies, among others.

The MESCP understands that this objectives cannot be achieved by the market,3 reason why the State intervenes as the resource allocator and is who has the task of conducting this model. Similarly, the model does not only focuses on building an industrialized Bolivia but also seeks to solve the poverty social problems, unemployment and low social mobility.

This model is designed for the Bolivian economy and its success is based on the good State administration of natural sources.

II. MATHEMATIC FORMALIZATION OF THE ECONOMIC SOCIAL COMMUNITARIAN PRODUCTIVE MODEL

a. Assumptions of the model

Given last framework in which the economy will perform. The model specification starts with the explanation of the following assumptions.[4]

1. There are only two sectors which generate the dynamic of the economy: one strategic sector (state sector) and an income and employment generating sector (also called value-added sector). The production of these sectors is represented by X_1 and X_2, respectively. The income and employment generating sector is administered by all other actors of the economy.[5]

   At the same time, it is important to note that $X_1(t)$ and $X_2(t)$ are time functions, but in order to avoid excess in notation, the subscript t or this functional form is not used.
2. Surpluses are generated in both sectors,[6] due to the fact that in both, the strategic sector and in the income and employment generating sector, there are productive activities. Consequently, according to Villegas and Aguirre (1989), it is assumed that:

$$X_1 = XN_1 + XE_1 \quad (1)$$

$$X_2 = XN_2 + XE_2 \quad (2)$$

   Where, XN_i is the necessary product of the sector i and XE_i is the surplus product of sector
   i, for all $i = \{1,2\}$.

3. The surplus of the strategic sector is exogenous, but decreasing and it is defined through the following function:

[3] The financial crisis that erupted in USA in mid-2008, and the fact that this crisis was originated in the financial market which is theoretically the closest to the definition of a perfect competence market, led to many economists to indicate that the crisis revealed the market failures and its consequences when it is not sufficiently regulated by the State.
[4] It should be clarified that this model is not within any theoretical framework of economic literature.
[5] The performers of the economy are: the State, private sector, social and cooperative economy and the communitarian economy.
[6] According to Villegas and Aguirre (1989) the economic surplus is total production minus the amount used to remunerate the work.

$$XE_1 = a_0 e^{-\gamma_1 t} \tag{3}$$

Where, $a_0, \gamma_1 \in R^+$ both are constants and $\frac{\dot{XE}_1}{XE_1} = -\gamma_1$

4. The strategic sector whose activity implicates the non-renewable resource exploitation must consider the resources restriction of the economy, namely: $\int_0^T r_t dt = R$. Where R is the natural resource quantity restriction of the country expressed in terms of production and r_t is the amount of natural resources extracted over time. By the Hotelling Rule[7] $P_t = P_0 e^{it}$ where P_t is the price of raw materials. The raw materials function demand is $P_t = \psi_0 - \psi_1 r_t$, where ψ_1 and $\psi_2 \in R^+$ are constants. Resolving the system, the duration time and the initial optimal price of extraction is found.[8]

5. The surplus of the income and employment generating sector which corresponds to the private sector, is exogenous and increasing, defined through the following function:

$$XE_2^p = b_0 e^{\beta_0 t} \tag{4}$$

Where both, $b_0, \beta_0 \in R^+$ are constants and $\frac{\dot{XE}_2^p}{XE_2^p} = \beta_0$

6. The surplus of the income and employment generating sector which belongs to the State increases at a constant rate, namely $\frac{\dot{XE}_2^e}{XE_2^e} = \sigma$.[9]

7. It is assumed that there is a definite function of the income and employment generating sector total surplus: $XE_2 = \left[XE_2^p \right]^{\tau_1} \left[XE_2^e \right]^{\tau_2}$, where $0 < \tau_1 < 1$ y $0 < \tau_2 < 1 \in R^+$ are constants. These parameters show the participation of each one in the total surplus. Meanwhile, applying neperian logarithm and taking the derivative over time:

$$\frac{\dot{XE}_2}{XE_2} = \tau_1 \beta_0 + \tau_2 \sigma = \beta_1$$

[7] See Hotelling (1931) for more details.
[8] It is important to note that this assumption enforces previous assumption.
[9] This parameter is assumed exogenous, as it is considered as endogenous, this would have to be a function of Government Spending, which would extremely complicate obtaining the analytical solution

8. Domestic demand is a determinant of economic growth in the model. Even though these variables vary over time, the participation of domestic demand in the product will be constant (both D(t) and Y(t) are time functions, but in order to avoid too many notations they will not be denoted). Such participation is presented by the following expression:

$$\frac{D}{Y} = d \tag{5}$$

Where, $d = cte \in R^+$ y $(0 < d < 1)$

9. Domestic demand will be superior or equal to the spending of the redistributing State, $D \geq G$, This spending G (t) is a time function and favors domestic demand through public investment, wages and conditional transfers, among others.
10. The State intervenes in the economy through government spending destined to both sectors, and through direct transfers to the population, in order to achieve its objectives of reducing poverty and improve income redistribution.[10]
11. In each period, government spending is financed by taxes on a share of the strategic sector surplus and on a share of the income and employment generating sector. Therefore, the State will always maintain a policy of balanced budget.
12. The share of the economy surplus as participation of the product will be constant.[11] Such participation will be represented by the following expression:

$$\frac{XE}{Y} = h \tag{6}$$

Where, $h = cte \in R^+$ y $(0 < h < 1)$

b. Development of the model

As mentioned in the assumptions 1 and 10, the State is concerned with the development of the production in each sector of the economy. The State sets as objectives to reach a production level X_1^* and X_2^* in each period of time. In order to simplify the analysis it is possible to assume that the effective production in each sector will be the same as the objective production, meaning that $X_1^* = X_1$ and $X_2^* = X_2$.

The greater each sector effective levels of production, or its equivalent, the greater the levels of each sector objective production, then the State receives greater levels of satisfaction. Therefore,

[10] It is important to mention that it is not the objective of this research to model economic agents (for example, households), so it is not needed a formal characterization of poverty and redistribution problems in the model.
[11] So it is needed to only identify which one of the actors in the economy appropriates it. In this model the State controls somehow the surplus, with taxes on the private sector or completely to dabble in the economy.

there is a utility function that represents the choices of the State regarding the production of each sector. This function is modelled as follows:

$$U\left(X_1, X_2\right) = X_1^{\alpha_1} X_2^{\alpha_2} \tag{7}$$

Where, α_1 y $\alpha_2 \in R^+$ are constants.

The State can influence each sector production through the spending. At the same time, as redistributing State, in order to finance these expenses it must obtain revenues, such revenues are collected by means of proportional tax on each sector added production.

In this framework, higher levels of spending implicates higher tax rates on the added production, which affect mainly the income and employment generating sector, which in turn creates disincentives in this sector. Thus, the State seeks to set optimally the production objectives of each sector subject to its revenues redistribution.[12]

At the same time, it must be noted that the strategic sector revenues comes from both taxes and the economic surplus that it generates. As it was mentioned in assumption 11, the State will maintain a policy of balanced budget so that the revenue is the same as the spending in each moment. Thus the redistributor State spending restriction will be given by the following expression:

$$G = \theta_1 X_1 + \theta_2 X_2 \tag{8}$$

Where $\theta_1 X_1 \equiv XE_1$ is the economic surplus generated by the strategic sector. Meanwhile $0 < \theta_1 < 1$ and $0 < \theta_2 < 1 \in R^+$, and $\theta_2 X_2 \equiv XE_2$ is the economic surplus generated by the income and employment generating sector that the State appropriates through the active participation. Therefore, the generated economic surplus, $XE_2 \geq \theta_2 X_2$, is greater or equal to what the State receives.

The optimization process of the State is resolved with Lagrange Method:

$$L = X_1^{\alpha_1} X_2^{\alpha_2} - \lambda\left(-G + \theta_1 X_1 + \theta_2 X_2\right) \tag{9}$$

Deriving (9) respect of X_1,

$$\frac{\partial L}{\partial X_1} = \alpha_1 X_1^{\alpha_1 - 1} X_2^{\alpha_2} - \lambda\theta_1 = 0 \tag{10}$$

[12] In the framework of the MESCP, the State captures a great share of the economy surplus, and uses it as instrument to impact both sectors through spending or through the active participation as entrepreneur. At the end, based on assumption 11, what is important is to find the production behavior dynamic equilibrium for both sectors in the economy.

Deriving (10) respect of X_2,

$$\frac{\partial L}{\partial X_2} = \alpha_2 X_1^{\alpha_1} X_2^{\alpha_2 - 1} - \lambda \theta_2 = 0 \tag{11}$$

The following relation is found, which is constant[13]

$$\frac{X_2}{X_1} = \frac{\theta_1}{\theta_2} \frac{\alpha_2}{\alpha_1} \tag{12}$$

On the other hand, the Redistributor State's revenues can be expressed as follows:

$$G \leq XE_1 + XE_2 \tag{13}$$

Finding the dynamic and applying assumptions 2, 3, 5, 6 and 7:

$$\frac{\dot{G}}{G} \leq -\frac{\gamma_1 \theta_1 X_1}{\theta_1 X_1 + \theta_2 X_2} + \frac{\beta_1 \theta_2 X_2}{\theta_1 X_1 + \theta_2 X_2} \tag{14}$$

Where, $u = \frac{\theta_1 X_1}{\theta_1 X_1 + \theta_2 X_2}$ and $v = \frac{\theta_2 X_2}{\theta_1 X_1 + \theta_2 X_2}$, are the surplus of the strategic sector and of the income and employment generating sector, respectively.

So, it can be written:

$$\frac{\dot{G}}{G} \leq -\gamma_1 u + \beta_1 v \tag{15}$$

Equation (15) suggests a negative impact from the strategic sector and a positive impact from the income and employment generating sector on the State spending.

The u dynamic is given by:[14]

$$\frac{\dot{u}}{u} = -a_3 + b_3 v - c_3 w \tag{16}$$

Applying the assumption 13, the dynamic of v is represented by:

$$\frac{\dot{v}}{v} = a_2 - b_2 u \tag{17}$$

[13] The optimum cannot be obtained because it must be remembered that these variables are in function of time, but if the relation which generated the constant parameters was used.
[14] See Appendix 1 for details.

Following the budget balanced assumption 9 and 11, the following dynamic arises:

$$\frac{\dot{D}}{D} \geq \frac{\dot{G}}{G} \tag{18}$$

Based on assumption 8 the results are that the domestic demand dynamic is as follows:

$$\frac{\dot{D}}{D} \leq \frac{\dot{Y}}{Y} \tag{19}$$

Meaning that: $\frac{\dot{Y}}{Y} \geq \frac{\dot{G}}{G}$

On the other hand, considering the income equality of the Redistributor State and finding the dynamics, the results are that even though the State revenue depends on the income and employment generating sector, also the growth rate of this sector will depend on the dynamic of Government spending.

$$\frac{\dot{X}_2}{X_2} = \left(1 + \frac{\alpha_1}{\alpha_2}\right)\left(\frac{\dot{G}}{G} + \frac{\alpha_1 \gamma_1}{\alpha_1 + \alpha_2}\right) \tag{20}$$

Using (18) one has:

$$\frac{\dot{X}_2}{X_2} \leq \left(1 + \frac{\alpha_1}{\alpha_2}\right)\left(\frac{\dot{D}}{D} + \frac{\alpha_1 \gamma_1}{\alpha_1 + \alpha_2}\right) \tag{21}$$

Considering that the income and employment generating sector is essential, because it includes manufacture (added value sector), then the following relation is considered:

$$w = \frac{X_2}{Y} \tag{22}$$

Which represents the participation of the income and employment generating sector in the product. Applying neperian logarithms and deriving:

$$\frac{\dot{w}}{w} = \frac{\dot{X}_2}{X_2} - \frac{\dot{Y}}{Y} \tag{23}$$

Replacing, sorting out and simplifying:

$$\frac{\dot{w}}{w} = -a_1 + b_1 v \tag{24}$$

Finally, the income and employment generating sector will depend of the income and employment generating sector's economic surplus.

Sorting out the equations to resolve, a good approximation is found of the mathematic super-predator model, while not the same logic:[15]

$$\begin{cases} \dot{v} = (a_2 - b_2 u)v \\ \dot{u} = (-a_3 + b_3 v - c_3 w)u \\ \dot{w} = -(a_1 - b_1 v)w \end{cases} \tag{25}$$

Finally, it is obtained the dynamic equation system (25), where the first line represents the dynamic surplus of the income and employment generating sector, the second line constitutes the strategic sector dynamic surplus, and the third equation expresses the dynamic of the income and employment generating sector participation in the product.

Solving the equation system a set of equilibrium points is found,[16] but only one is considered, that is $X(t) = \left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, -\frac{a_3}{c_3} + \frac{b_3 a_1}{c_3 b_1} \right)$, which allows the interaction of the three analyzed variables.[17] The equilibrium of the solution is obtained by the Jacobian matrix:

$$J = \begin{pmatrix} 0 & -\frac{b_2 a_1}{b_1} & 0 \\ \frac{b_3 a_2}{b_2} & 0 & -\frac{c_3 a_2}{b_2} \\ \frac{b_1 a_3}{c_3} - \frac{b_3 a_1}{c_3} & 0 & 0 \end{pmatrix}$$

Its own values are: $\lambda_2 = -\frac{a_1 a_2 b_3}{3 b_1 A} + A$ y $\lambda_{1,3} = \frac{a_1 a_2 b_3}{6 b_2 A} - \frac{A}{2} \pm \frac{\sqrt{3}\left(\frac{a_1 a_2 b_3}{3 b_1 A} + A \right) i}{2}$ where

[15] It is extremely important to note that the parameters of the equation system are generated from the added variables and since the set of actors in the economy are not modelled from a microeconomic point of view, thus this mathematical model is found within the macroeconomic models that have fondness to the Lucas critique.

[16] Equilibrium solutions are: $(0,0,0)$, $\left(\frac{a_3}{b_3}, \frac{a_2}{b_2}, 0 \right)$ y $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, -\frac{a_3}{c_3} + \frac{b_3 a_1}{c_3 b_1} \right)$.

[17] See Appendix 1 for more details.

$$A=\left(\left(\frac{\left(\frac{b_3 a_1 a_2}{b_1}\right)^3}{27}+\frac{\left(a_1 a_2\left(\frac{a_1 b_3}{b_1}-a_3\right)\right)^2}{4}\right)^{\frac{1}{2}}-\frac{a_1 a_2\left(\frac{b_3 a_1}{b_1}-a_3\right)}{2}\right)^{\frac{1}{3}}$$

Therefore, the solutions will be sinusoidal functions, which will fluctuate in time.

c. Model Calibration

For model calibration the following parameter assumptions of the equation system are considered:[18]

- $\alpha_1=\alpha_2$, which means that the State preferences regarding each sector production is the same.
- $\tau_1=\tau_2$, which means that the participation of the State and private sector in the income and employment generating sector is the same.
- $\theta_2=h$, which means that the impact of the State's share collected from sector X_2 is equal to the surplus participation of the product.

With these three parameter assumptions one has:

$$a_1=b_3=\gamma_1$$

$$a_2=\tau\left(\beta_0+\sigma\right)+\frac{1}{2}\gamma_1$$

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[18]
$$\begin{cases} a_1=\gamma_1 \\ b_1=\gamma_1\dfrac{\alpha_1}{\alpha_2}-\tau_1\beta_0-\tau_2\sigma=\gamma_1-\tau\left(\beta_0+\sigma\right) \\ a_2=\tau_1\beta_0+\tau_2\sigma+\gamma_1\dfrac{\alpha_1}{\alpha_1+\alpha_2}=\tau\left(\beta_0+\sigma\right)+\dfrac{1}{2}\gamma_1 \\ b_2=\left(\tau_1\beta_0+\tau_2\sigma\right)\theta_2\dfrac{1}{h}\dfrac{h}{\theta_2}\dfrac{\alpha_1}{\alpha_2}=\left(\tau_1\beta_0+\tau_2\sigma\right)\dfrac{\alpha_1}{\alpha_2}=\tau\left(\beta_0+\sigma\right) \\ a_3=-\gamma_1\dfrac{\alpha_1}{\alpha_2}=-\gamma_1 \\ b_3=\gamma_1\dfrac{\alpha_1}{\alpha_2}=\gamma_1 \\ c_3=\left(\tau_1\beta_0+\tau_2\sigma\right)\dfrac{\alpha_1}{\alpha_2}\dfrac{\theta_2}{h}\dfrac{\alpha_1}{\alpha_2}=\left(\tau_1\beta_0+\tau_2\sigma\right)\left(\dfrac{\alpha_1}{\alpha_2}\right)^2\dfrac{\theta_2}{h}=\tau\left(\beta_0+\sigma\right) \end{cases}$$

$$a_3 = -\gamma_1$$

$$b_1 = \gamma_1 - \tau(\beta_0 + \sigma)$$

$$b_2 = c_3 = \tau(\beta_0 + \sigma)$$

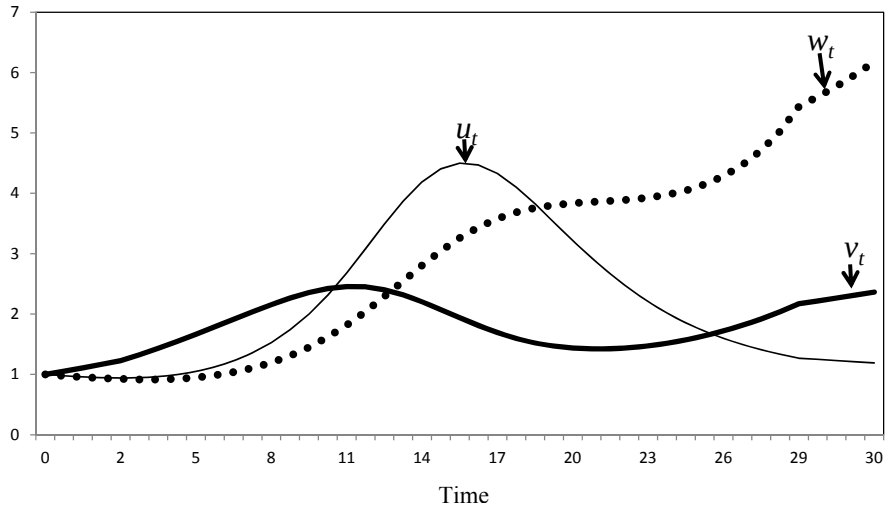
Finally, considering that the equation system is based on four essential parameters:

Table 1: Essential Parameters

Parameter	Description	Value
γ_1	Growth rate of the strategic surplus sector.	0.2
β_0	Growth rate of the income and employment generating sector.	0.1
τ	State participation on the surplus generated in the income and employment generating sector.	0.5
σ	Growth rate of the State surplus (a constant defined and administrated by the redistributor State.	0.01

Performing a small simulation of the equation system, assigning values to the essential parameters[19] and finding the dynamic solution[20], the resulting dynamics are:

Graph 1: Dynamic Solution of the Economic Social Community Productive Model



[19] The parameters of the model are approximated values that allow to find a numeric solution of the model results. Nevertheless, it must be noted that the results are sensible to parameter variations, being able to easily change the presented dynamics.
[20] See Appendix 1

The first thick line represents the evolution of the income and employment generating sector surplus. The thin line shows the evolution of the strategic sector surplus.

The dotted line represents the participation of the added value in the product. The simulation based on the essential parameters shows that the growth of the income and employment generating sector has an expansion and contraction cycle but it is not fluctuating. While the strategic sector surplus is increasing up to certain extent in time due to the increase in exploitation, followed by a descendent pattern, due to the limitation of the natural resources. In this framework, highlights the importance of the added value generating sector participation to promote a sustained growth in the economy. It act as a new engine that boosts the economy when the natural resources are depleted and the surplus of the strategic sector diminishes.

d. Stability of the equilibrium points

It is necessary to test if the solutions found for the differential equation model are stable. Deriving the equation system respect to v, u y w:[21]

$$Df = \begin{pmatrix} a_2 - b_2 u & -b_2 v & 0 \\ b_3 u & -a_3 + b_3 v - c_3 w & -c_3 u \\ -b_1 w & 0 & -a_1 + b_1 v \end{pmatrix}$$

Fort the equilibrium point $(0,0,0)$ and $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, -\frac{a_3}{c_3} + \frac{b_3 a_1}{c_3 b_1}\right)$ it was found that being the real part of the eigenvalues negative,[22] it is concluded that the equilibrium point is unstable in time. This implicates that the equilibrium changes in time, without arriving to a stationary state.

By contrast, the equilibrium point $\left(\frac{a_3}{b_3}, \frac{a_2}{b_2}, 0\right)$ is locally (neutrally) stable[23], which implicates that in time an optimum stationary state can be reached. This reflects that in that moment in time the

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[21] $x(t) = (v(t), u(t), w(t))$ denoted as a variable vector.

[22] Theorem: Let X* be an equilibrium point of $\dot{x} = f(x)$, with $f : R^n \to R^n$ being class C^1. Then: If $Df(x^*)$ has all the eigenvalues λ_j with real part $Re(\lambda_j) < 0 \to x^*$ is locally asymptomatically stable;

If $\exists \lambda_j$ with real part $Re(\lambda_j) > 0 \to x^*$ is unstable;

If $Re(\lambda_j) \leq 0 \ \forall_j$ y $\exists \lambda_j$ with $Re(\lambda_j) = 0 \to$ nothing can be said

[23] It is worth to note that for the second equilibrium point, nothing can be said until it is proven with Liapunov functions. Establishing a Liapunov function $V := a_2 b_3 v + a_3 b_2 u - a_3 a_2 ln(vu) + w^2$, vector gradient: $\nabla V\left(\frac{a_3}{b_3}, \frac{a_2}{b_2}, 0\right) = (0,0,0)$, furthermore,

economy will be in its efficiency point, it is worth to note that it is a local equilibrium point and not a total one.

In conclusion, the mathematic model developed in this document reaches a stationary state, where the economy will be at its optimum. This is accomplished by using the government economic surplus of the Redistributor State, who initially capture the surplus of the strategic sector and later captures the part of the income and employment generating sector economic surplus. This equilibrium point is considered an efficient point.

III. CONCLUSION

After more than 20 years of the neoliberal model validity, which placed an excessive emphasis on the market autoregulation ability and denied the structural characteristics of Bolivia, then the Economic Social Communitarian Productive Model pose a new vision on the Bolivian economy performance linked proposals (a set of policies) to generate high rates of economic growth in a medium and long term.

The Economic Social Communitarian Productive Model identifies the existence of two large sectors in the economy: a strategic sector and income and employment generating sector. Furthermore, the importance of the State is recognized in order to reach an adequate distribution of the resources and avoid the formation of monopolies in the economy. Given the characteristics of the strategic sector, this will be controlled by the State, so, the surplus of the strategic sector shall be transferred to the income and employment generating sector favoring its participation and the industrialization of the country. The great potential of domestic demand will allow developing this sector, in order to make it competitive with respect to the rest of the world; and in this way reaching a sustained economic growth.

$$\nabla^2 V = \begin{pmatrix} \frac{a_3 a_2}{v^2} & 0 & 0 \\ 0 & \frac{a_3 a_2}{u^2} & 0 \\ 0 & 0 & 2 \end{pmatrix} \rightarrow \nabla^2 V\left(\frac{a_3}{b_3}, \frac{a_2}{b_2}, 0\right) = \begin{pmatrix} \frac{a_3 a_2}{\left(\frac{a_3}{b_3}\right)^2} & 0 & 0 \\ 0 & \frac{a_3 a_2}{\left(\frac{a_2}{b_2}\right)^2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Is a positive defined matrix then V reaches a strict minimum local in $\left(\frac{a_3}{b_3}, \frac{a_2}{b_2}, 0\right)$.

Lastly, $\frac{d}{dt} V\left(x(t)\right) = \nabla V\left(x(t)\right) * \dot{x}(t) = 0$

$\therefore \left(\frac{a_3}{b_3}, \frac{a_2}{b_2}, 0\right)$ is locally (neutrally) stable.

The mathematical model generated a system of differential equations type Lokta-Voltera whose numerical solution shows an oscillatory constant growth of the income and employment generating sector and an increase of the economic surplus at increasing rates. Clearly, the mathematical model shows the theoretical possibility of generating sustained growth in economies of this type.

In this way, the Economic Social Communitarian Productive Model visualizes the income and employment generating sector as the main sector that can generate sustained growth in the economy. Due to the strategic sector being controlled by the Redistributor State, it uses the economic surplus to industrialize and expand the productive bases of the country, action which in turn has a considerable impact in the income and employment generating sector. Another part of this surplus is relocated towards the private sector through improvements of domestic demand (through the transfer policies, higher wages policies, etc.), what drives the domestic market and generates movement in the economy allowing the income and employment generating sector to develop.

While the domestic demand stays unsatisfied it will bring sustained growth. In the long term when the income and employment generating sector has enough capacity to cover the domestic demand, this sector will become the new engine for the Bolivian economy.

Finally, the existence of an equilibrium point of stationary state implicates that the Economic Social Communitarian Productive Model can reach an optimal state if the State intervenes in an active and permanent way in the economy through the transfer of resources from one sector to the other.

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APPENDIX 1. MATHEMATICAL DEMONSTRATIONS

Model Demonstration

Starting from the function,

$$U\left(X_1, X_2\right) = X_1^{\alpha_1} X_2^{\alpha_2} \tag{1}$$

Subject to a restriction,

$$G = \theta_1 X_1 + \theta_2 X_2 \tag{2}$$

The multiplying Lagrange Method, using (1) and (2),

$$L = X_1^{\alpha_1} X_2^{\alpha_2} - \lambda\left(-G + \theta_1 X_1 + \theta_2 X_2\right) \tag{3}$$

From equation (3) deriving respect to X_1 and equaling to zero,

$$\frac{\partial L}{\partial X_1} = \alpha_1 X_1^{\alpha_1 - 1} X_2^{\alpha_2} - \lambda\theta_1 = 0 \tag{4}$$

From equation (3) deriving respect to X_2 and equaling to zero,

$$\frac{\partial L}{\partial X_2} = \alpha_2 X_1^{\alpha_1} X_2^{\alpha_2 - 1} - \lambda\theta_2 = 0 \tag{5}$$

Solving the equation system (4) and (5) the relationship of ambos sector was found.

$$\frac{X_2}{X_1} = \frac{\alpha_2}{\alpha_1}\frac{\theta_1}{\theta_2} \tag{6}$$

Using the assumptions 6 and 7 given that $XE_1 = a_0 e^{-\gamma_1 t}$ and $XE_2 = \left[XE_2^p\right]^{\tau_1}\left[XE_2^e\right]^{\tau_2}$, where the dynamics are: $\frac{\dot{X}\,E_1}{XE_1} = -\gamma_1$ y $\frac{\dot{X}\,E_2}{XE_2} = \tau_1\beta_0 + \tau_2\sigma = \beta_1$

It is also important: $XE_1 \equiv \theta_1 X_1$ and $XE_2 \le \theta_2 X_2$, consequently and using equation (2)

$$G \le \theta_1 X_1 + \theta_2 X_2 \equiv XE_1 + XE_2 \tag{7}$$

Finding the dynamic in time of equation (7),

$$\dot{G} \le \dot{XE}_1 + \dot{XE}_2 \tag{8}$$

Dividing by G equation (8),

$$\frac{\dot{G}}{G} \le \frac{\dot{XE}_1}{G} + \frac{\dot{XE}_2}{G} \tag{9}$$

Using a mathematical artifice in (9),

$$\frac{\dot{G}}{G} \leq \frac{\frac{\dot{XE}_1}{XE_1}}{\frac{G}{XE_1}} + \frac{\frac{\dot{XE}_2}{XE_2}}{\frac{G}{XE_2}} = \frac{-\gamma_1}{G} XE_1 + \frac{\beta_1}{G} XE_2 = \\ -\gamma_1 \frac{XE_1}{XE_1 + XE_2} + \beta_1 \frac{XE_2}{XE_1 + XE_2} = -\gamma_1 u + \beta_1 v \tag{10}$$

Finding the dynamic,

$$u = \frac{XE_1}{XE_1 + XE_2} = \frac{XE_1}{G} \tag{11}$$

Applying logarithms to (11),

$$ln(u) = ln(XE_1) - ln(G) \tag{12}$$

Deriving in time (12),

$$\frac{\dot{u}}{u} = \frac{\dot{XE}_1}{XE_1} - \frac{\dot{G}}{G} \tag{13}$$

Replacing (6) and (11), using mathematical artifices,

$$\frac{\dot{u}}{u} = -\gamma_1 - (-\gamma_1 u + \beta_1 v) = -\gamma_1 + \gamma_1 \frac{XE_1}{XE_1 + XE_2} - \beta_1 v = \\ -\gamma_1 + \gamma_1 \frac{XE_1 XE_2}{(XE_1 + XE_2) XE_2} - \beta_1 v = -\gamma_1 + \frac{XE_1}{XE_2} v - \beta_1 v = \\ -\gamma_1 + \gamma_1 \frac{\theta_1 X_1}{\theta_2 X_2} v - \beta_1 v = -\gamma_1 + \gamma_1 \frac{\alpha_1}{\alpha_2} v - \beta_1 v = -\underbrace{\gamma_1}_{f_1} + \left(\underbrace{\gamma_1 \frac{\alpha_1}{\alpha_2} - \beta_1}_{g_1} \right) v \tag{14}$$

Finally,

$$\dot{u} = (-f_1 + g_1 v) u = -(f_1 - g_1 v) u \tag{15}$$

Finding the dynamic,

$$v = \frac{XE_2}{G} \tag{16}$$

Applying logarithm to (16),

$$ln(v) = ln(XE_2) - ln(G) \tag{17}$$

Deriving in time (17),

$$\frac{\dot{v}}{v} = \frac{\dot{XE}_2}{XE_2} - \frac{\dot{G}}{G} \tag{18}$$

Replacing (6) and $w = \frac{X_2}{Y}$, with some mathematical artifices,

$$\begin{aligned}
&\frac{\dot{v}}{v} = \beta_1 - (-\gamma_1 u + \beta_1 v) = \beta_1 - \left(-\gamma_1 u + \beta_1 v \frac{Y}{Y}\right) = \beta_1 - \left(-\gamma_1 \frac{XE_1}{XE_1 + XE_2} + \beta_1 \frac{XE_2}{XE_1 + XE_2}\frac{Y}{Y}\right) = \\
&\beta_1 - \left(-\gamma_1 \frac{1}{1 + \frac{XE_2}{XE_1}} + \beta_1 \frac{\theta_2 X_2}{XE}\frac{Y}{Y}\right) = \beta_1 - \left(-\gamma_1 \frac{1}{1 + \frac{XE_2}{XE_1}} + \beta_1 \frac{\theta_2 Y}{XE} w\right) = \beta_1 - \left(-\gamma_1 \frac{1}{1 + \frac{\alpha_2}{\alpha_1}} + \beta_1 \frac{\theta_2 Y}{XE} w\right) = \\
&\beta_1 - \left(-\gamma_1 \frac{\alpha_1}{\alpha_1 + \alpha_2} + \beta_1 \theta_2 \frac{1}{h} w\right) = \underbrace{\beta_1 + \gamma_1 \frac{\alpha_1}{\alpha_1 + \alpha_2}}_{f_2} - \underbrace{\beta_1 \theta_2 \frac{1}{h}}_{g_2} w
\end{aligned} \tag{19}$$

Finally from (19),

$$\dot{v} = (f_2 - g_2 w) v \tag{20}$$

In conclusion finding the dynamic for w

$$w = \frac{X_2}{Y} \tag{21}$$

Applying neperian logarithm and deriving in time equation (21),

$$\frac{\dot{w}}{w} = \frac{\dot{X}_2}{X_2} - \frac{\dot{Y}}{Y} \tag{22}$$

Again, starting from G, with some changes

$$G = XE_1 + \theta_2 X_2 \tag{23}$$

Deriving in time the equation (23),

$$\dot{G} = \dot{XE}_1 + \theta_2 \dot{X}_2 \tag{24}$$

Dividing between G the equation (24) and performing mathematical artifices

$$\frac{\dot{G}}{G}=\frac{\frac{\dot{XE}_1}{XE_1}}{\frac{G}{XE_1}}+\frac{\theta_2\frac{\dot{X}_2}{X_2}}{\frac{G}{X_2}}=\frac{-\gamma_1\alpha_1}{\alpha_1+\alpha_2}+\frac{\theta_2}{\frac{\theta_1X_1}{X_2}+\frac{\theta_2X_2}{X_2}}\frac{\dot{X}_2}{X_2}=\frac{-\gamma_1\alpha_1}{\alpha_1+\alpha_2}+\frac{\theta_2}{\frac{\theta_1\theta_2\alpha_1}{\theta_1\alpha_2}+\theta_2}\frac{\dot{X}_2}{X_2}=$$
$$\frac{-\gamma_1\alpha_1}{\alpha_1+\alpha_2}+\frac{\theta_2}{\frac{\theta_2\alpha_1}{\alpha_2}+\theta_2}\frac{\dot{X}_2}{X_2}=\frac{-\gamma_1\alpha_1}{\alpha_1+\alpha_2}+\frac{1}{\frac{\alpha_1}{\alpha_2}+1}\frac{\dot{X}_2}{X_2}=\frac{-\gamma_1\alpha_1}{\alpha_1+\alpha_2}+\frac{\alpha_2}{\alpha_1+\alpha_2}\frac{\dot{X}_2}{X_2} \quad (25)$$

Clearing X_2 from equation (25),

$$\frac{\dot{X}_2}{X_2}=\frac{\alpha_1+\alpha_2}{\alpha_2}\left(\frac{\gamma_1\alpha_1}{\alpha_1+\alpha_2}+\frac{\dot{G}}{G}\right)=\frac{\alpha_1+\alpha_2}{\alpha_2}\left(\frac{\gamma_1\alpha_1}{\alpha_1+\alpha_2}+\frac{\dot{G}}{G}\right) \quad (26)$$

Following the reasoning $D\geq G$ y $\frac{D}{Y}=d$ there is that

$$\frac{\dot{D}}{D}\geq\frac{\dot{G}}{G} \text{ y } \frac{\dot{D}}{D}=\frac{\dot{Y}}{Y},$$

Consequently $\frac{\dot{Y}}{Y}\geq\frac{\dot{G}}{G}$ or it is the same,

$$\frac{\dot{G}}{G}\leq\frac{\dot{Y}}{Y} \quad (27)$$

Considering the looseness of certain unevenness, it is resolved for the parity

Replacing X_2 and Y of equations (26) y (27),

$$\frac{\dot{w}}{w}=\left(1+\frac{\alpha_1}{\alpha_2}\right)\left(\frac{\dot{G}}{G}+\frac{\alpha_1\gamma_1}{\alpha_1+\alpha_2}\right)-\frac{\dot{G}}{G}=\frac{\alpha_1+\alpha_2}{\alpha_2}\frac{\dot{G}}{G}+\frac{\gamma_1\alpha_1}{\alpha_2}-\frac{\dot{G}}{G}=\frac{\gamma_1\alpha_1}{\alpha_2}+\frac{\alpha_1}{\alpha_2}\frac{\dot{G}}{G}=$$
$$\frac{\gamma_1\alpha_1}{\alpha_2}+\frac{\alpha_1}{\alpha_2}\left(-\gamma_1u+\beta_1v\right)=\underset{\uparrow\\ -f_3}{\frac{\gamma_1\alpha_1}{\alpha_2}}-\underset{\uparrow\\ g_3}{\gamma_1\frac{\alpha_1}{\alpha_2}}u+\underset{\uparrow\\ h_3}{\beta_1\frac{\alpha_1}{\alpha_2}}v \quad (28)$$

Finally equation (28),

$$\frac{\dot{w}}{w}=-f_3+g_3v-h_3u \quad (29)$$

Then from equation (29),

$$\dot{w} = \left(-a_3 + b_3 v - c_3 u\right) w \tag{30}$$

The dynamic equation system comprises equations (20), (30) and (15), respectively,

$$\begin{cases} \dot{v} = \left(f_2 - g_2 w\right) v \\ \dot{w} = \left(-f_3 + g_3 v - h_3 u\right) w \\ \dot{u} = -\left(f_1 - g_1 v\right) u \end{cases} \tag{31}$$

Additionally, it can be demonstrated that,

$$w = \frac{X_2}{Y} = \frac{\theta_2 X_2}{\theta_2 Y} \frac{XE_1}{XE} \frac{XE}{XE_1} = \frac{h\alpha_2}{\theta_2 \alpha_1} u \rightarrow \frac{\theta_2}{h} \frac{\alpha_1}{\alpha_2} w = u$$

Applying logarithms and deriving in time,

$$\frac{\dot{w}}{w} = \frac{\dot{u}}{u}$$

The system is transformed in,

$$\begin{cases} \dot{v} = \left(\underbrace{f_2}_{a_2} - \underbrace{g_2 \frac{h\alpha_2}{\theta_2 \alpha_1}}_{b_2} u \right) v \\ \dot{u} = \left(-\underbrace{f_3}_{a_3} + \underbrace{g_3}_{b_3} v - \underbrace{h_3 \frac{\theta_2}{h} \frac{\alpha_1}{\alpha_2}}_{c_3} w \right) u \\ \dot{w} = -\left(\underbrace{f_1}_{a_1} - \underbrace{g_1}_{b_1} v \right) w \end{cases} \tag{32}$$

Transforming variables conveniently,

$$\begin{cases} \dot{v} = (a_2 - b_2 u) v \\ \dot{u} = (-a_3 + b_3 v - c_3 w) u \\ \dot{w} = -(a_1 - b_1 v) w \end{cases} \tag{33}$$

Resolving the dynamic equation system (33), finding the solutions of the homogenous system,

$$\begin{cases} (a_2 - b_2 u) v = 0 \\ (-a_3 + b_3 v - c_3 w) u = 0 \\ -(a_1 - b_1 v) w = 0 \end{cases} \tag{34}$$

Resolving by substitution, for (34),

$$a_2 - b_2 u = 0 \rightarrow u = \frac{a_2}{b_2} \tag{35}$$

And,

$$-a_1 + b_1 v = 0 \rightarrow v = \frac{a_1}{b_1} \tag{36}$$

Replacing (35) in (36),

$$-a_3 + b_3 \frac{a_1}{b_1} = c_3 w \rightarrow w = \frac{-a_3}{c_3} + \frac{a_1 b_3}{b_1 c_3} \tag{37}$$

Being that it is a system of superior level and according to the fundamental algebra theorem, three solutions are found.

The equilibrium points are: $(0,0,0)$, $\left(\frac{a_3}{b_3}, \frac{a_2}{b_2}, 0\right)$ and $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, -\frac{a_3}{c_3} + \frac{b_3 a_1}{c_3 b_1}\right)$

Deriving from the equation system (34), (35) and (36), there is the following Jacobian matrix.

$$Df = \begin{pmatrix} a_2 - b_2 u & -b_2 v & 0 \\ b_3 u & -a_3 + b_3 v - c_3 w & -c_3 u \\ -b_1 w & 0 & -a_1 + b_1 v \end{pmatrix} \tag{38}$$

Replacing for $(0,0,0)$ in (38),

$$Df(0,0,0) = \begin{pmatrix} a_2 & 0 & 0 \\ 0 & -a_3 & 0 \\ 0 & 0 & -a_1 \end{pmatrix} \tag{39}$$

Replacing for $\left(\frac{a_3}{b_3},\frac{a_2}{b_2},0\right)$ in (38),

$$Df\left(\frac{a_3}{b_3},\frac{a_2}{b_2},0\right)=\begin{pmatrix}0 & -\frac{b_2a_3}{b_3} & 0\\ \frac{b_3a_2}{b_2} & 0 & -\frac{c_3a_2}{b_2}\\ 0 & 0 & -a_1+\frac{b_1a_3}{b_3}\end{pmatrix} \tag{40}$$

Replacing for $\left(\frac{a_1}{b_1},\frac{a_2}{b_2},-\frac{a_3}{c_3}+\frac{b_3a_1}{c_3b_1}\right)$ in (38),

$$Df\left(\frac{a_1}{b_1},\frac{a_2}{b_2},-\frac{a_3}{c_3}+\frac{b_3a_1}{c_3b_1}\right)=\begin{pmatrix}0 & -\frac{b_2a_1}{b_1} & 0\\ \frac{b_3a_2}{b_2} & 0 & -\frac{c_3a_2}{b_2}\\ \frac{b_1a_3}{c_3}-\frac{b_3a_1}{c_3} & 0 & 0\end{pmatrix} \tag{41}$$

For each equilibrium point the solution of the dynamic system can be found,

The solution for $(0,0,0)$

$$v(t)=c_1e^{a_2t},\ w(t)=c_2e^{-a_3t},\ u(t)=c_3e^{-a_1t} \tag{42}$$

The solution for $\left(\frac{a_1}{b_1},\frac{a_2}{b_2},-\frac{a_3}{c_3}+\frac{b_3a_1}{c_3b_1}\right)$,

First its own values are found which are:

$\lambda_1=-\frac{a_1a_2b_3}{3b_1A}+A$ y $\lambda_{2,3}=\frac{a_1a_2b_3}{6b_1A}-\frac{A}{2}\pm\frac{\sqrt{3}\left(\frac{a_1a_2b_3}{3b_1A}+A\right)i}{2}$ donde

$$A=\left(\left(\frac{\left(\frac{b_3a_1a_2}{b_1}\right)^3}{27}+\frac{\left(a_1a_2\left(\frac{b_3c_3}{b_1}-a_3\right)\right)^2}{4}\right)^{\frac{1}{2}}-\frac{a_1a_2\left(\frac{b_3c_3}{b_1}-a_3\right)}{2}\right)^{\frac{1}{3}}$$

To make the work easier, the following notation was used, where Z is

$$Z=\frac{\sqrt{3}\left(\left(\frac{\left(a_2a_1^2b_3-a_1a_2a_3b_1\right)^2}{4b_1^2}+\frac{a_1^3a_2^3b_3^3}{27b_1}\right)^{\frac{1}{2}}-\frac{a_2a_1^2b_3-a_1a_2a_3b_1}{2b_1}\right)^{\frac{1}{3}}+\frac{a_1a_2b_3}{3b_1\left(\left(\frac{\left(a_2a_1^2b_3-a_1a_2a_3b_1\right)^2}{4b_1^2}+\frac{a_1^3a_2^3b_3^3}{27b_1}\right)^{\frac{1}{2}}-\frac{a_2a_1^2b_3-a_1a_2a_3b_1}{2b_1}\right)^{\frac{1}{3}}}i}{2} \quad (43)$$

Besides *X*

$$X=-\frac{\left(\left(\frac{\left(a_2a_1^2b_3-a_1a_2a_3b_1\right)^2}{4b_1^2}+\frac{a_1^3a_2^3b_3^3}{27b_1}\right)^{\frac{1}{2}}-\frac{a_2a_1^2b_3-a_1a_2a_3b_1}{2b_1}\right)^{\frac{1}{3}}}{2}+\frac{a_1a_2b_3}{6b_1\left(\left(\frac{\left(a_2a_1^2b_3-a_1a_2a_3b_1\right)^2}{4b_1^2}+\frac{a_1^3a_2^3b_3^3}{27b_1}\right)^{\frac{1}{2}}-\frac{a_2a_1^2b_3-a_1a_2a_3b_1}{2b_1}\right)^{\frac{1}{3}}} \quad (44)$$

The matrix of own values using equation (43) and (44),

$$\begin{pmatrix} X+Zi & 0 & 0 \\ 0 & Z & 0 \\ 0 & 0 & X-Zi \end{pmatrix} \quad (45)$$

The eigenvectors are:

$$V_1=\begin{pmatrix} -\frac{c_3\left(X+Zi\right)}{a_1b_3-a_3b_1} \\ \frac{b_1c_3\left(X+Zi\right)^2}{a_1^2b_2b_3-a_1a_3b_1b_2} \\ 1 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} -\dfrac{c_3 Z}{a_1 b_3 - a_3 b_1} \\ \dfrac{b_1 c_3 \left(Z\right)^2}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2} \\ 1 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} \dfrac{c_3 \left(X + Zi\right)}{a_1 b_3 - a_3 b_1} \\ \dfrac{b_1 c_3 \left(X + Zi\right)^2}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2} \\ 1 \end{pmatrix} \tag{46}$$

Being complex solutions they were worked for the real part and the imaginary part,

$$Re\left\{Ve^{\lambda t}\right\},\ Im\left\{Ve^{\lambda t}\right\} \tag{47}$$

For V_1, the solution was performed applying the Euler Formula,

$$Re\left\{V_1 e^{\lambda t}\right\} = Re\left\{\begin{pmatrix} -\dfrac{c_3\left(X+Zi\right)}{a_1 b_3 - a_3 b_1} \\ \dfrac{b_1 c_3\left(X+Zi\right)^2}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2} \\ 1 \end{pmatrix} e^{(X+Zi)t}\right\} =$$

$$e^{Xt} Re\left\{\begin{pmatrix} -\dfrac{c_3\left(X+Zi\right)}{a_1 b_3 - a_3 b_1} \\ \dfrac{b_1 c_3\left(X+Zi\right)^2}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2} \\ 1 \end{pmatrix} \left(cos\left(Zt\right) + isen\left(Zt\right)\right)\right\} =$$

$$e^{Xt}\begin{pmatrix} \dfrac{-c_3 X}{a_1 b_3 - a_3 b_1}\left(cos\left(Zt\right) + sen\left(Zt\right)\right) \\ \dfrac{b_1 c_3}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2}\left(X^2 cos\left(Zt\right) - Z^2 cos\left(Zt\right) - 2XZsen\left(Zt\right)\right) \\ cos\left(Zt\right) \end{pmatrix} \tag{48}$$

$$Im\left\{V_1 e^{\lambda t}\right\} = Im\left\{\begin{pmatrix} -\dfrac{c_3\left(X+Zi\right)}{a_1b_3-a_3b_1} \\ \dfrac{b_1c_3\left(X+Zi\right)^2}{a_1^2b_2b_3-a_1a_3b_1b_2} \\ 1 \end{pmatrix} e^{(X+Zi)t}\right\} =$$

$$e^{Xt} Im\left\{\begin{pmatrix} -\dfrac{c_3\left(X+Zi\right)}{a_1b_3-a_3b_1} \\ \dfrac{b_1c_3\left(X+Zi\right)^2}{a_1^2b_2b_3-a_1a_3b_1b_2} \\ 1 \end{pmatrix} \left(cos\left(Zt\right)+isen\left(Zt\right)\right)\right\} =$$

$$e^{Xt}\begin{pmatrix} \dfrac{-c_3Z}{a_1b_3-a_3b_1}\left(cos\left(Zt\right)+sen\left(Zt\right)\right) \\ \dfrac{b_1c_3}{a_1^2b_2b_3-a_1a_3b_1b_2}\left(2XZcos\left(Zt\right)-X^2sen\left(Zt\right)-Z^2sen\left(Zt\right)\right) \\ sen\left(Zt\right) \end{pmatrix} \tag{49}$$

For V_2

$$Re\left\{V_2 e^{\lambda t}\right\} = Re\left\{\begin{pmatrix} -\dfrac{c_3 Z}{a_1 b_3 - a_3 b_1} \\ \dfrac{b_1 c_3 \left(Z\right)^2}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2} \\ 1 \end{pmatrix} e^{Zt}\right\} =$$

$$e^{Zt} Re\left\{\begin{pmatrix} -\dfrac{c_3 Z}{a_1 b_3 - a_3 b_1} \\ \dfrac{b_1 c_3 \left(Z\right)^2}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2} \\ 1 \end{pmatrix}\right\} =$$

$$e^{Xt} \begin{pmatrix} -\dfrac{c_3 Z}{a_1 b_3 - a_3 b_1} \\ \dfrac{b_1 c_3 \left(Z\right)^2}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2} \\ 1 \end{pmatrix} \tag{50}$$

For V_3

$$Re\left\{V_3 e^{\lambda t}\right\} = Re\left\{\begin{pmatrix} -\dfrac{c_3\left(X - Zi\right)}{a_1 b_3 - a_3 b_1} \\ \dfrac{b_1 c_3\left(X - Zi\right)^2}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2} \\ 1 \end{pmatrix} e^{(X - Zi)t}\right\} =$$

$$e^{Xt} Re\left\{\begin{pmatrix} -\dfrac{c_3\left(X - Zi\right)}{a_1 b_3 - a_3 b_1} \\ \dfrac{b_1 c_3\left(X - Zi\right)^2}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2} \\ 1 \end{pmatrix} \left(cos\left(Zt\right) - isen\left(Zt\right)\right)\right\} =$$

$$e^{Xt} \begin{pmatrix} \dfrac{c_3}{a_1 b_3 - a_3 b_1}\left(-Xcos\left(Zt\right) + Zsen\left(Zt\right)\right) \\ \dfrac{b_1 c_3}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2}\left(X^2 cos\left(Zt\right) - Z^2 cos\left(Zt\right) - 2XZsen\left(Zt\right)\right) \\ cos\left(Zt\right) \end{pmatrix} \tag{51}$$

$$Im\left\{V_3 e^{\lambda t}\right\} = Im\left\{\begin{pmatrix} -\dfrac{c_3\left(X-Zi\right)}{a_1 b_3 - a_3 b_1} \\ \dfrac{b_1 c_3\left(X-Zi\right)^2}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2} \\ 1 \end{pmatrix} e^{(X-Zi)t}\right\} =$$

$$e^{Xt} Im\left\{\begin{pmatrix} -\dfrac{c_3\left(X-Zi\right)}{a_1 b_3 - a_3 b_1} \\ \dfrac{b_1 c_3\left(X-Zi\right)^2}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2} \\ 1 \end{pmatrix} \left(cos\left(Zt\right) - isen\left(Zt\right)\right)\right\} =$$

$$e^{Xt}\begin{pmatrix} \dfrac{c_3}{a_1 b_3 - a_3 b_1}\left(Zcos\left(Zt\right) - Xsen\left(Zt\right)\right) \\ \dfrac{b_1 c_3}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2}\left(-2XZcos\left(Zt\right) - X^2 sen\left(Zt\right) + Z^2 sen\left(Zt\right)\right) \\ sen\left(Zt\right) \end{pmatrix} \tag{52}$$

Finally the solutions of the dynamic equation systems are:

$$\begin{pmatrix} v(t) \\ w(t) \\ u(t) \end{pmatrix} = e^{Xt} \left[C_1 \begin{pmatrix} \frac{-c_3 X}{a_1 b_3 - a_3 b_1}\left(\cos(Zt) + sen(Zt)\right) \\ \frac{b_1 c_3}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2}\left(X^2 \cos(Zt) - Z^2 \cos(Zt) - 2XZsen(Zt)\right) \\ \cos(Zt) \end{pmatrix} + \right.$$

$$C_2 \begin{pmatrix} \frac{-c_3 Z}{a_1 b_3 - a_3 b_1}\left(\cos(Zt) + sen(Zt)\right) \\ \frac{b_1 c_3}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2}\left(2XZ\cos(Zt) - X^2 sen(Zt) - Z^2 sen(Zt)\right) \\ sen(Zt) \end{pmatrix} +$$

$$C_3 \begin{pmatrix} -\frac{c_3 Z}{a_1 b_3 - a_3 b_1} \\ \frac{b_1 c_3 (Z)^2}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2} \\ 1 \end{pmatrix} +$$

$$C_4 \begin{pmatrix} \frac{c_3}{a_1 b_3 - a_3 b_1}\left(-X\cos(Zt) + Zsen(Zt)\right) \\ \frac{b_1 c_3}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2}\left(X^2 \cos(Zt) - Z^2 \cos(Zt) - 2XZsen(Zt)\right) \\ \cos(Zt) \end{pmatrix} +$$

$$\left. C_5 \begin{pmatrix} \frac{c_3}{a_1 b_3 - a_3 b_1}\left(Z\cos(Zt) - Xsen(Zt)\right) \\ \frac{b_1 c_3}{a_1^2 b_2 b_3 - a_1 a_3 b_1 b_2}\left(-2XZ\cos(Zt) - X^2 sen(Zt) + Z^2 sen(Zt)\right) \\ sen(Zt) \end{pmatrix} \right]$$

Where C_1, C_2, C_3, C_4 y C_5 are constants.

APPENDIX 2. MATLAB PROGRAMME

```
function nmodecoscp
% Differential Equation System
function xdot = vdpol(t,x)
xdot=zeros(3,1);
xdot(1)=0.155.*x(1)-0.055.*x(2).*x(1);
xdot(2)=-.2.*x(2)+0.2.*x(1).*x(2)-.055.*x(2).*x(3);
xdot(3)=-0.2.*x(3)+0.145.*x(1).*x(3);
fin
t0=0; tf=30;
x0=[1 1 1]';
[t,x]=ode45(@vdpol,[t0 tf],x0);
t,x
% Graph
hFig=figure(1);
set(hFig,'Position',[500 400 900 500])
plot(t,x);
hline=findobj(gcf,'type','line');
set(hline(1),'LineWidth',1.5,'LineStyle',':')
set(hline(3),'LineWidth',2)
legend_h=legend('XXX','YYY','ZZZ','Location','Best');
set(legend_h,'Interpreter','latex','string',{'Surplus from the income and employment generating sector
(${v_t}$)','Surplus of the strategic sector estrat$\acute{e}$gico (${u_t}$)','Participaci$\acute{o}in the
income and employment generating sector in the product{w_t}$)'});
xlabel('tiempo (t)')
fin
```